NAG Toolbox for MATLAB

d01gc

1 Purpose

d01gc calculates an approximation to a definite integral in up to 20 dimensions, using the Korobov–Conroy number theoretic method.

2 Syntax

```
[vk, res, err, ifail] = d01gc(f, region, npts, vk, nrand, 'ndim', ndim,
'itrans', itrans)
```

3 Description

d01gc calculates an approximation to the integral

$$I = \int_{c_1}^{d_1} dx_1, \ldots, \int_{c_n}^{d_n} dx_n \qquad f(x_1, x_2, \ldots, x_n) \tag{1}$$

using the Korobov–Conroy number theoretic method (see Korobov 1957, Korobov 1963 and Conroy 1967). The region of integration defined in (1) is such that generally c_i and d_i may be functions of $x_1, x_2, \ldots, x_{i-1}$, for $i = 2, 3, \ldots, n$, with c_1 and d_1 constants. The integral is first of all transformed to an integral over the n-cube $[0, 1]^n$ by the change of variables

$$x_i = c_i + (d_i - c_i)y_i, \qquad i = 1, 2, \ldots, n.$$

The method then uses as its basis the number theoretic formula for the n-cube, $[0, 1]^n$:

$$\int_0^1 dx_1 \cdots \int_0^1 dx_n g(x_1, x_2, \ldots, x_n) = \frac{1}{p}\sum_{k=1}^{p} g\left(\left\{k\frac{a_1}{p}\right\}, \ldots, \left\{k\frac{a_n}{p}\right\}\right) - E \tag{2}$$

where $\{x\}$ denotes the fractional part of x, $a_1, a_2, \ldots, a_n$ are the so-called optimal coefficients, E is the error, and p is a prime integer. (It is strictly only necessary that p be relatively prime to all $a_1, a_2, \ldots, a_n$ and is in fact chosen to be even for some cases in Conroy 1967.) The method makes use of properties of the Fourier expansion of $g(x_1, x_2, \ldots, x_n)$ which is assumed to have some degree of periodicity. Depending on the choice of $a_1, a_2, \ldots, a_n$ the contributions from certain groups of Fourier coefficients are eliminated from the error, E. Korobov shows that $a_1, a_2, \ldots, a_n$ can be chosen so that the error satisfies

$$E \leq CKp^{-\alpha}\ln^{\alpha\beta}p \tag{3}$$

where α and C are real numbers depending on the convergence rate of the Fourier series, β is a constant depending on n, and K is a constant depending on α and n. There are a number of procedures for calculating these optimal coefficients. Korobov imposes the constraint that

$$a_1 = 1 \qquad \text{and} \qquad a_i = a^{i-1}(\mathrm{mod}\, p) \tag{4}$$

and gives a procedure for calculating the parameter, a, to satisfy the optimal conditions.

In this function the periodisation is achieved by the simple transformation

$$x_i = y_i^2(3 - 2y_i), \qquad i = 1, 2, \ldots, n.$$

More sophisticated periodisation procedures are available but in practice the degree of periodisation does not appear to be a critical requirement of the method.

An easily calculable error estimate is not available apart from repetition with an increasing sequence of values of p which can yield erratic results. The difficulties have been studied by Cranley and Patterson 1976 who have proposed a Monte Carlo error estimate arising from converting (2) into a stochastic integration rule by the inclusion of a random origin shift which leaves the form of the error (3) unchanged; i.e., in the formula (2), $\left\{k\frac{a_i}{p}\right\}$ is replaced by $\left\{\alpha_i + k\frac{a_i}{p}\right\}$, for $i = 1, 2, \ldots, n$, where each α_i, is uniformly distributed over $[0, 1]$. Computing the integral for each of a sequence of random vectors α allows a 'standard error' to be estimated.

This function provides built-in sets of optimal coefficients, corresponding to six different values of p. Alternatively, the optimal coefficients may be supplied by you. Functions d01gy and d01gz compute the optimal coefficients for the cases where p is a prime number or p is a product of two primes, respectively.

4 References

Conroy H 1967 Molecular Shroedinger equation VIII. A new method for evaluting multi-dimensional integrals *J. Chem. Phys.* **47** 5307–5318

Cranley R and Patterson T N L 1976 Randomisation of number theoretic methods for mulitple integration *SIAM J. Numer. Anal.* **13** 904–914

Korobov N M 1957 The approximate calculation of multiple integrals using number theoretic methods *Dokl. Acad. Nauk SSSR* **115** 1062–1065

Korobov N M 1963 *Number Theoretic Methods in Approximate Analysis* Fizmatgiz, Moscow

5 Parameters

5.1 Compulsory Input Parameters

1: **f – string containing name of m-file**

f must return the value of the integrand f at a given point.

Its specification is:

```
[result] = f(ndim, x)
```

Input Parameters

1: **ndim – int32 scalar**

n, the number of dimensions of the integral.

2: **x(ndim) – double array**

The co-ordinates of the point at which the integrand f must be evaluated.

Output Parameters

1: **result – double scalar**

The result of the function.

2: **region – string containing name of m-file**

region must evaluate the limits of integration in any dimension.

Its specification is:

```
[c, d] = region(ndim, x, j)
```

Input Parameters

1: **ndim – int32 scalar**

n, the number of dimensions of the integral.

2: **x(ndim) – double array**

$\mathbf{x}(1), \ldots, \mathbf{x}(j-1)$ contain the current values of the first $(j-1)$ variables, which may be used if necessary in calculating c_j and d_j.

3: **j – int32 scalar**

The index j for which the limits of the range of integration are required.

Output Parameters

1: **c – double scalar**

The lower limit c_j of the range of x_j.

2: **d – double scalar**

The upper limit d_j of the range of x_j.

3: **npts – int32 scalar**

The Korobov rule to be used. There are two alternatives depending on the value of **npts**.

(i) $1 \leq$ **npts** ≤ 6.

In this case one of six preset rules is chosen using 2129, 5003, 10007, 20011, 40009 or 80021 points depending on the respective value of **npts** being 1, 2, 3, 4, 5 or 6.

(ii) **npts** > 6.

npts is the number of actual points to be used with corresponding optimal coefficients supplied in the array **vk**.

Constraint: **npts** ≥ 1.

4: **vk(ndim) – double array**

If **npts** > 6, **vk** must contain the n optimal coefficients (which may be calculated using d01gy or d01gz).

If **npts** ≤ 6, **vk** need not be set.

5: **nrand – int32 scalar**

The number of random samples to be generated in the error estimation (generally a small value, say 3 to 5, is sufficient). The total number of integrand evaluations will be **nrand** $\times$ **npts**.

Constraint: **nrand** ≥ 1.

5.2 Optional Input Parameters

1: **ndim – int32 scalar**

Default: The dimension of the array **vk**.

n, the number of dimensions of the integral.

Constraint: $1 \leq$ **ndim** ≤ 20.

2: **itrans – int32 scalar**

Indicates whether the periodising transformation is to be used.

itrans $= 0$

The transformation is to be used.

itrans $\neq 0$

The transformation is to be suppressed (to cover cases where the integrand may already be periodic or where you want to specify a particular transformation in the definition of user-supplied real function **f**).

Suggested value: **itrans** $= 0$.

Default: 0

5.3 Input Parameters Omitted from the MATLAB Interface

None.

5.4 Output Parameters

1: **vk(ndim) – double array**

If **npts** > 6, **vk** is unchanged.

If **npts** ≤ 6, **vk** contains the n optimal coefficients used by the preset rule.

2: **res – double scalar**

The approximation to the integral I.

3: **err – double scalar**

The standard error as computed from **nrand** sample values. If **nrand** $= 1$, then **err** contains zero.

4: **ifail – int32 scalar**

0 unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

ifail $= 1$

On entry, **ndim** < 1,
or **ndim** > 20.

ifail $= 2$

On entry, **npts** < 1.

ifail $= 3$

On entry, **nrand** < 1.

7 Accuracy

An estimate of the absolute standard error is given by the value, on exit, of **err**.

8 Further Comments

The time taken by d01gc will be approximately proportional to **nrand** $\times p$, where p is the number of points used.

The exact values of **res** and **err** returned by the function will depend (within statistical limits) on the sequence of random numbers generated within the function by calls to g05ka. To ensure that the results returned by d01gc in separate runs are identical, you should call g05kb immediately before calling d01gc; to ensure that they are different, call g05kc.

9 Example

```
d01gc_f.m

function result = f(ndim,x)
  result = cos(0.5+2*sum(x)-double(ndim));
```

```
d01gc_region.m

function [c,d] = region(ndim, x, j)
  c=0;
  d=1;
```

```
npts = int32(2);
vk = zeros(4,1);
nrand = int32(4);
[vkOut, res, err, ifail] = d01gc('d01gc_f', 'd01gc_region', npts, vk,
nrand)
```

```
vkOut =
           1
         792
        1889
         191
res =
    0.4400
err =
   1.7550e-06
ifail =
           0
```